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MATHEMATICS IN BUSINESS AND INDUSTRY.

BY H. T. H. PIAGGIO, D.Sc.

AT the last meeting of the Mathematical Association Dr. W. P. Milne read a paper on "The Industrialisation of Mathematics," in which he asked that the Association should try to get into closer touch with the industrial applications of our subject. In the ensuing discussion some speakers expressed their doubts whether any mathematics beyond the merest elements really came into industry at all. Now I have been making enquiries into this question for several years, and perhaps the results may be of interest to readers of the *Gazette*.

The persons with whom I discussed this subject may be divided into two classes, financial and manufacturing. It must be admitted that the knowledge of mathematics possessed by the first group was not large, and further that even this small amount was so rusty that they were unable to apply it themselves to their professions. For instance, an accountant said that there was one branch of mathematics which he considered very useful indeed, but unfortunately he had forgotten it so completely that he could not recall even its name! (It turned out that he was referring to Logarithms.) He further said that he obtained all his results on compound interest and annuities from tables. His chief practical difficulty was to know whether to add or subtract in complicated cases under the Excess Profits Tax where a deduction had to be made from another deduction which itself had to be diminished by yet another deduction . . . and so on. This is rather like the problem of removing a series of brackets with minus signs before each.

However, several men had enough idea of mathematics to know that it *could* solve certain of their business problems, although they personally could not do the work. A bank clerk asked me to work out what sum of money would have to be put away now to meet certain liabilities several years hence. This was, of course, a perfectly straightforward question on compound interest. The only difficulty was that the result was required to the nearest penny, while the amount ran into several thousands of pounds, so eight-figure logarithms were required.

A more interesting problem was that propounded by a bank manager. He said that the commonest mistake of a cashier was to enter such a sum as 6/10 as £6 10s. 0d. Of course, when the books were made up at the end of the day the cash appeared to be £6 3s. 2d. short. His difficulty was, knowing the amount of the apparent deficiency and assuming it to be due to a single

mistake of this kind, to find the amount entered wrongly. He was delighted when he was supplied with a formula (or rather two: there being two cases to consider) to do this, and he said that it would save him a lot of worry. This problem might be given to a Matriculation class.

Mathematicians who try to help their business friends will find that they are expected to know everything and give solutions almost immediately. An insurance official whose duty it was to value haystacks was able to calculate the volume of a stack from the measurements he took, but he did not know how many pounds of hay went to the cubic foot. He asked me for the value of this constant, and when he found that I didn't know it, he caustically remarked, "I see that your knowledge is only theoretical and not practical." But he was mollified when the information was obtained for him a few days later.

One of the most remarkable men in the Midlands is a financier who applies the ideas (though not the formal methods) of the mathematical theory of Probability to business speculations. He displays marvellous acumen in computing, almost instinctively, the probable value of legacies left to one member of a family provided they survive several other members, and of similarly complicated assets. It should be stated that this man was top of his class in mathematics when he was at school, although he did not get beyond elementary algebra and geometry. For many years he offered an annual prize, consisting of a gold medal and a situation in his office to the best mathematician in the elementary schools of the town.

As an exceptional example of the direct use of mathematics in business may be mentioned the case of a teacher whose elderly relative was thinking of buying an annuity from an insurance company. It was an easy matter to consult the tables of expectation of life, and then to work out what yearly income should be expected, assuming the rate of interest to be that given by War Bonds. The income offered by the insurance company proved to be very much less than this amount. A little consideration of the varying dates of redemption of War Savings Certificates, War Bonds and Loans, and certain Colonial Government Stocks showed how an annuity scheme could be worked by the family, giving about 50 per cent. more than was offered by the insurance company, with other advantages as well. The scheme was put into operation, and appears to be working very satisfactorily.

We will now consider the second group of persons, namely those connected with manufacturing. There is a great disparity in the degree to which mathematics is used between different industries, and even between different branches of the same industry. In engineering, for example, there are still firms where it is looked upon as a great feat to be able to calculate the change-wheels required to cut a thread of given pitch on a screw-cutting lathe. One small firm in London bought a new lathe of a rather complicated kind which was accompanied by a book of directions. They were quite unable to use the machine for a time, because two wheels had to be set at a certain angle given by a formula involving trigonometrical ratios and logarithms. On the other hand, a firm like Parsons' Turbine Co. will conduct elaborate researches, employing Cambridge Wranglers and other highly trained men. These researches are so thoroughly carried out that the Admiralty has now come to rely upon them, and to show its trust by paying for the estimated additions to the guaranteed speeds of its destroyers, the estimates being based on these researches. A former member of their technical staff said that the work was very interesting to a mathematician, except that it was rather trying at first to have to realise that the answers had to be absolutely correct, and that no credit was given merely for a correct method!

This employment of mathematicians by industrial firms is not very general as yet, but it seems to be growing. An old student wrote recently to say that he had just been appointed head of the research department of one of the best

known Rubber companies. His efforts to trace the cause of failure of solid tyres had led him to a differential equation, which is not solved in any text-book, and is so complicated that it might defy even Dr. Brodetáky and all his array of graphical weapons. It involves that peculiar function $E(x)$, the greatest integer contained in x. It is strange that a function that has hitherto been regarded as purely artificial, invented merely to puzzle the unfortunate student of the theory of limits and continuity, should now turn up in a real physical problem of commercial interest. But surprises of this kind are constantly occurring. Even geometry finds its application. In some researches on explosives carried out by a chemist a certain vapour-pressure curve was plotted. A mathematical colleague recognised it as approximately a cubic, and calculated its equation. By taking the differential coefficient of the function it was possible to use a formula of physical chemistry and obtain a measure of the explosive power. Who would have thought that cubics had anything to do with tri-nitro-toluene? Another application of geometry was required when a shell factory had to make gauges for a new kind of shell. This needed a little coordinate geometry and the use of very accurate tables of squares and square roots.

An N.C.O. in charge of computers engaged in calibrating artillery wrote asking for information about some points in the theory of errors and their application to his work. It turned out that the answers to his questions were contained in some notes prepared for an agricultural college which was analysing the statistics of egg-laying!

Since the war British Optical manufacturers have made great efforts to improve their instruments, and they are employing a few mathematicians to work out suitable combinations of lenses. Unfortunately the designer of optical instruments finds that the ordinary University text-books on Optics have hardly any application to the problems presented by actual practice. Academic Optics bears about the same relation to Technical Optics as the picturesque pipe-playing shepherd of poetry to the genuine farm-hand.

It is interesting to know that the use of the slide rule appears to be growing. At one time it was little known outside the engineer's drawing-office, but lately it has been used by analytical chemists and others to calculate percentages. It is probable that a knowledge of the use of the slide rule has been spread by the artillery cadet schools. But this factor must not be overestimated. Those members of the association who have instructed artillery cadets, and still more those who have talked with full-fledged and experienced gunnery officers, will know that the courage which sustained our countrymen against the fiercest onslaughts of a brutal foe was yet insufficient to overcome the terrors of a harmless-looking slide rule. A class of cadets was being shown how to calculate the battery angle, given the battery range, the observer's range, and the exterior angle. After several unsuccessful attempts to solve an example of this problem, one cadet gloomily said, "I hope this war won't last much longer!"

The officers who had seen active service generally relied entirely upon certain rules-of-thumb, such as "Angle of Sight in minutes equals difference of height in inches divided by range in hundreds of yards," or "If the exterior angle is less than 80 degrees, add 2 to the first figure of the angle and take that percentage of the true base, and then take this as the base in the range-finding formula." It will be an interesting exercise for mathematicians to identify the problem in the solution of triangles to which the last rule relates.

Of course there were a few artillery officers whose knowledge of mathematics was very thorough indeed. One of the most distinguished of the younger Cambridge mathematicians worked out the trajectory of our anti-aircraft guns, and several others were engaged on more or less mathematical work at Woolwich Arsenal. Moreover, it is due to artillery officers that the subject of Nomography (a graphical method of dealing with functions of two or more

independent variables) has been brought into prominence. Nomography was developed, chiefly by M. D'Ocagne, as much as thirty-five years ago, but it was little known in England until quite recently. Another subject greatly developed by the war is the theory of aviation. The general principles of stability of flight were laid down by Professor Bryan before the war, but these have been greatly developed and applied to practical designing by Bairstow and others at the National Physical Laboratory. Their efforts have resulted in the production of inherently stable aeroplanes. Meanwhile, the more theoretical investigations have been continued by Bryan and Brodetsky.

Let us now sum up and try to draw a few conclusions. Industry seems to be more and more recognising its need of mathematics. This need will be chiefly supplied by a small number of highly-trained experts, generally employed by very large firms, to whom the salary required is only a minute proportion of their total expenses. But in addition to these experts there must be a very large number of men who, though unable to solve mathematical problems themselves, can realise in what cases the assistance of an expert is likely to be useful. The education of the experts cannot be too thorough. It is extremely hazardous to say that any given branch of mathematics will fail to find an application. It is extremely important that the mathematics should be accompanied by some experimental physics. It is also necessary that all the latest devices for the numerical or graphical treatment of algebraical and differential equations should be mastered. The mathematics that is restricted to problems that come out nicely by elegant methods is as much out of touch with the realities of to-day as the beautiful but helpless young ladies of the early Victorian era, ready to faint at the slightest emergency. The expert of to-day must be prepared to deal with any problem and to carry the work to the bitter end, however tedious it may be. Of course the problem may often be reduced to manageable proportions by adopting certain reasonable approximations.

The task of recognising what approximations are allowable will sometimes be very difficult. It will be necessary to consider carefully the degree of accuracy required in the particular industry. The expert who does not lubricate his academic machinery with a large amount of the oil of common-sense may find that it is impossible to grind out any results at all.

As for the other and larger class who are merely to recognise when to call in the expert, it is unnecessary to say much here, as most of the efforts of the Mathematical Association are devoted to the question of the required "Democratisation of Mathematics." For the majority it does not matter very much what details are taught, as long as the general impression is clear and definite. They should be made to realise that Mathematics is a subject of enormous power, capable of application to many aspects of life. Progress in this direction is certainly being made. A newspaper which is said to maintain its mammoth circulation by giving the public what they want, presented them lately with an elaborate graphical argument concerning the influenza epidemic. This graph appeared on the same page as the pictures of a royal wedding. Who will now despair of the possibility of "bringing Mathematics home to the hearts of the people"!

H. T. H. PIAGGIO.

GLEANINGS FAR AND NEAR.

26. The Intrusion of Mathematics. The Governing Body of the Union shall be a Council elected by the branches. Each branch shall elect to the Council a number of members equal to the integer nearest to n/s, where n is the number of members of the branch and s is a number to be fixed from time to time by the Council. Zero shall not be counted as an integer. If n/s is of the form $p+\frac{1}{2}$, where p is an integer, it shall be taken as equal to $p+1$, according as p is odd or even.—*National Union of Scientific Workers.* Proposed Rule 8 (election of governing body), 1919.

EXTENSION OF THE THEORY OF INVERSION TO CONICS.

By The Rev. J. J. Milne, M.A.

[*The references are to the author's treatise on Cross-Ratio Geometry.*]

Let t be the centre of a circle, radius r, p any point in its plane, and draw the diameter through p. Then the inverse of p is a point p' on tp such that $tp \cdot tp' = r^2$. This circle is called the circle of inversion, its centre t the pole of inversion and the square of its radius the constant of inversion, and each of the points p, p' is the inverse of the other. This relation is a metric one: a descriptive relation can be obtained as follows.

Every point on the polar of p is conjugate to p, and the inverse of p is that conjugate point which lies on the diameter through p; in other words, the inverse of a point is the intersection of its polar with the diameter through the point.

As long as we take a circle as our curve of inversion it is immaterial which of these relations we take as our base. Suppose, however, we take a conic as the fundamental curve of inversion, then it is evident that we cannot deal with the metrical relation, but there is nothing to prevent us from considering the question descriptively.

Now, two of the distinguishing features of a circle considered as a member of the family of conics are, (1) that it passes through the two circular points at infinity, usually denoted by i, i', and (2) that its centre is the pole of the line at infinity. In passing from the circle to the conic these imaginary points will be replaced by real points I, I', lying on the conic, and t, the centre of the circle, will become T, the pole of the chord II'.

As orthogonal circles enter largely into the theory of inversion, it may be as well to consider what they become in the case of conics. Two orthogonal circles possess the following fundamental properties, from any one of which the others may be derived:

(1) The tangents at a point of intersection are at right angles.

(2) The tangents at a point of intersection pass through the centres.

(3) The centre of each circle is the pole of the common chord for the other circle and the pole of ii' for its own circle.

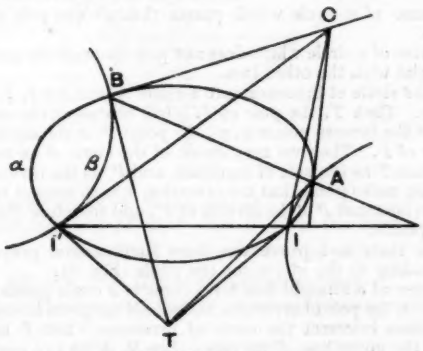

Fig. 1.

Now, let α, β be the two corresponding conics having II', AB for a pair of common chords, and let T, C be the poles of II' (Fig. 1).

Then

(1) The tangents at A divide II' harmonically.

(2) The tangents at A pass through T, C.

(3) T, the pole of II' for α, is the pole of AB for β.

From (1), these conics may be called harmotomic; these and other important properties relating to them are proved directly in *Cross-Ratio Geometry*, Cap. XIII.

When the curve of inversion is a circle there are three fundamental propositions, viz. (Fig. 2):

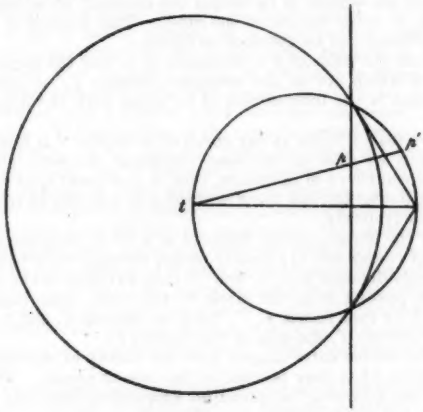

FIG. 2.

(1) The inverse of a line is a circle which passes through the pole of inversion, the pole of the line, the points i, i', and the points where the line cuts the circle of inversion.

(2) The inverse of a circle which passes through the pole of inversion is a straight line.

(3) The inverse of a circle which does not pass through the pole of inversion is a circle coaxial with the other two.

Transform the circle of inversion into a conic A, and let I, I' represent the circular points. Then T, the pole of II', will represent the centre t, and if P, P' represent the inverse points p, p', the point P' is the intersection of TP with the polar of P. Then we may speak of the conic A as the conic of inversion, the point T as the pole of inversion, and P' as the inverse of P for the point T, it being understood that the inversion is with respect to the conic A.

Note.—It is obvious that P is the inverse of P', and therefore P and P' may be called inverse points.

We will now state and prove the three fundamental properties for the conic corresponding to the above for the circle (Fig. 3):

(1) The inverse of a straight line for a conic is a conic passing through six known points, viz. the pole of inversion, the pole of the given line, and the points where their polars intersect the conic of inversion. Let T be the pole of inversion, DE the given line, F its pole. Let H, K be two conjugate points on DE, and let TH meet FK in P. Thus P, H are inverse points. Then, since the range (H) = the range (K),

$$\therefore \text{ the pencil } T(H) = \text{the pencil } F(K).$$

$\therefore$ the locus of P is a conic through T and F. The conic evidently also passes through D, E, I, I'. Another proof of this is given in *C.-R.* Art. 191.

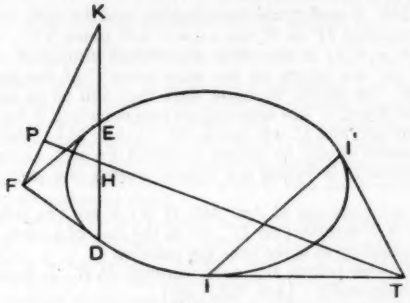

FIG. 3.

(2) The inverse of a conic passing through the pole of inversion T and the points I, I' is a straight line, viz. the other common chord corresponding to II'. This is obvious from (1), since P and H are inverse points.

(3) If A is the conic and T the pole of inversion, the inverse of a conic B which passes through I and I' but not through T is a conic C of the pencil determined by A and B. Let P be any point on B (Fig. 4), II' one of the

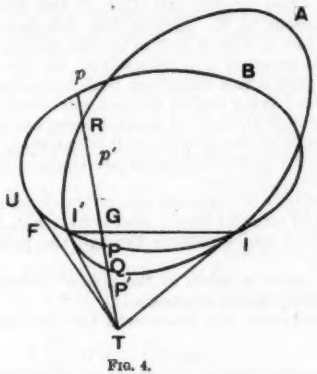

FIG. 4.

common chords of A and B, T the pole of II' for A. Join TP, meeting A in Q, R, B again in p, and II' in G. Let P', p' be the inverses of P, p.

Then $$(PP'QR) = -1 = (pp'QR) = (pp'RQ).$$

$\therefore$ by Art. 105, $(Pp, P'p', QR)$ is an involution range.

$\therefore$ by Sturm's Theorem, Art. 216, P', p' are points on a conic C of the pencil determined by A and B.

Since $$(TGQR) = -1 = (PP'QR) = (pp'QR),$$

$$\therefore (PP', pp', TG) \text{ is an involution range.}$$

$$\therefore (TGPp') = (GTP'p) = (TGpP') = \lambda,$$

and if the five points P, P', p, p', G are given, the sixth T is given, and is unique. $\therefore$ T is the centre, II' the axis and λ the constant of homology of B and C.

$\therefore$ by Art. 243, B and C are homologous, and by Art. 244, if TU is the tangent to B, meeting II' in F, the conic C will touch TU.

If $TP_1'Q_1P_1G_1p_1R_1p_1'$ is any other transversal through T, we still have to show that P_1, p_1' are points on the same conic C of the pencil as that on which P', p' lie. As above, we know that they will lie on a conic C' of the pencil, and that B and C' are homologous conics having T for centre and II' for axis of homology, and C' will touch TU by Art. 244. But only one conic of the pencil besides B can be drawn to touch TU; $\therefore$ C' coincides with C.

Cor. 1. The pole of inversion is a tangent vertex of a conic and its inverse conic.

Cor. 2. By the converse of Art. 248, if M, N are the poles of II' for B and C, Mp and NP' intersect on II', *i.e.* in the case of a circle and its inverse the radii corresponding to these lines are parallel.

If P, P' are inverse points, any conic through P, P', I, I' cuts the conic of inversion harmotomically. (Art. 205, Cor.)

Any conic B can be inverted into itself by taking as the conic of inversion a conic A which cuts it harmotomically, and if A is the conic of inversion and a conic B is inverted into itself, A and B are harmotomic.

Any two conics having the same common chord with the conic of inversion and cutting it harmotomically can be inverted into themselves. Let α, α' be any two conics, II', QR a pair of common chords, on QR take any point T, join TI and TI', draw TA a tangent to α. Then the conic β passing through A and touching TI, TI' at I, I' will be the conic of inversion and will be harmotomic to α. And by Art. 205, Cor., Q and R being conjugate points for β, α' is harmotomic to β. And by Art. 205, any transversal through T cuts each of the conics α, α' in a pair of inverse points. Q.E.D.

If two conics are inverted into themselves, the line joining their poles for II' inverts into a conic which cuts them both harmotomically.

A system of conics having double contact will invert into a pencil of conics, the pole of inversion and the pole of contact being two vertices of the common self-conjugate triangle of the pencil; and conversely, if a pencil of conics is inverted from a vertex of the self-conjugate triangle it inverts into a system of conics having double contact.

If we take as the pole of inversion one of the common points of the pencil, we obtain a pencil of straight lines; and conversely, the inverse of a pencil of lines is a pencil of conics passing through the pole of inversion and the centre of the pencil.

If any arbitrary point is taken as the pole of inversion, a pencil of conics will invert into another pencil of conics.

The theory of inversion was discovered for the circle in 1842 by Stubbs and Ingram.

JOHN J. MILNE.

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27. *E.* Dr. Whewell has just published *another* book . . . on education, in which he cries up the geometrical processes in preference to analysis.—Sir John Herschel to Mrs. Somerville.—M. S.

28. *F.* Mr. Adams told Somerville that the following sentence in the sixth edition of the *Connexion of the Physical Sciences*, published in the year 1842, put it into his head to calculate the orbit of Neptune. "If after the lapse of years the tables formed from a combination of numerous observations should still be inadequate to represent the motions of Uranus, the discrepancies may reveal the existence, nay, even the mass and orbit of a body placed for ever beyond the sphere of vision."—M. S.

NOTES FOR LESSONS INTRODUCTORY TO DIFFERENTIAL GEOMETRY.

By Eric H. Neville.

It is a commonplace of writers on differential geometry that the arc element of a curve is essentially positive, and there is universal agreement that the curvature of a curve in space also is positive.* That these two conventions are sometimes troublesome nobody † will deny; the object of the following paragraphs is to shew that they are neither necessary nor desirable even in the earliest stages, and that in fact the subject can be developed, smoothly and rapidly, without either of them.

The essence of the treatment I adopt is in the definition of a curve. The usual view is that a curve *acquires* the order of its points from the parameter by means of which it is studied. I assert that on the contrary *the curve of kinematical geometry is not the mere aggregate of its points, but is this collection of points with a definite serial order that has nothing to do with any particular variable used in discussing the curve.* Of any two distinct points of a kinematical curve, one *precedes* the other.

If P, Q are two distinct points of a curve, the secant PQ is not the Euclidean line through P and Q but the ray or directed line on which P and Q have the same order as on the curve: if P precedes Q on the curve, the secant PQ is the ray through P and Q whose direction is from P to Q, and the secant QP is this same ray. The tangent at a point O also is a ray, the limit of the secant PQ as P and Q approach O independently.

The chord PQ has its length measured along the secant; the length of PQ is therefore positive or negative according as P precedes or follows Q, and the length of QP is the negative of the length of PQ.

Analysis begins when the positions of a current point O on the curve are correlated with the values of a real variable t. If Q lies between P and R, then t_Q lies between t_P and t_R; hence for a given correlation the sign of $t_Q - t_P$ depends only on whether P precedes or follows Q. The sign of $t_Q - t_P$ when P precedes Q is called the sign of the correlation, and if this sign is positive, t is said to be an increasing or a positive variable for the curve.

The length of the chord PQ divided by $t_Q - t_P$ is a definite number with the sign of the correlation; on a regular curve this number tends to a unique limit as P and Q tend in any manner to O, and this limit, which will be denoted by w, is the speed of O with respect to t. If T is a second variable for the same curve, and W is the speed with respect to T,

$$W = \lim\{PQ/(T_Q - T_P)\} = \lim\{PQ/(t_Q - t_P)\}\{(t_Q - t_P)/(T_Q - T_P)\} = w\,dt/dT,$$

and therefore

$$\int_{T_P}^{T_Q} W\,dt = \int_{t_P}^{t_Q} w\,dt;$$

the value of the integral, which is thus shewn to be independent of the particular variable, defines the arc of the curve from P to Q. Since w has the sign of the correlation, the arc from P to Q is positive or negative according as P precedes or follows Q. The arc to the current point O from some definite point is denoted by s; the speed w is the derivative ds/dt, and since the arc is always found, in practice as well as in theory, by integration of the

* See for example Bell, pp. 277, 283, 288; Eisenhart, pp. 10, 14, 181; Forsyth, pp. 2, 4. Continental authors make the same assumptions.

† The difficulties are faced scrupulously by Eisenhart, as for example on pp. 118, 182. There are of course writers careless enough to break without suspicion conventions by which they profess to be bound.

speed, v is sometimes called the arc function of the curve for the variable t. By the adoption of the arc for variable, the speed can be made unity.

If OT is the tangent at O to a curve Γ and Ω is a fixed point, the point T at unit distance from Ω in the direction OT traces the tangential indicatrix of Γ. It is assumed that this indicatrix has no singularities and is itself an ordered curve, but the order of the points on the indicatrix is not made dependent on the order of the points on the original curve: if P precedes Q on Γ, the tangential image of P may either precede or follow the tangential image of Q on the indicatrix.

Since Ω is a fixed point, the motion of ΩT takes place in the plane through Ω containing the tangent to the curve described by T, and since O moves in the direction OT (possibly with a negative speed), the motion of OT is in the parallel plane through O. This plane through O, with a cyclic direction yet to be assigned, is the osculating plane, and the principal normal, a normal in this plane, is therefore parallel to the tangent to the indicatrix. Hence it is possible to define the direction of the principal normal to the original curve and the direction of the tangent to the tangential indicatrix to be the same; in effect, sometimes the order of the points on the indicatrix is known and determines the standard direction OP of the principal normal, but often it is the latter direction which is known and serves to distinguish the standard direction of the tangent to the indicatrix, and in this way to determine the order of precedence on the indicatrix.

The ratio of the speed of T to the speed of O is independent of the variable t and measures the curvature of the path of O; in other words, the curvature, which is denoted by κ, is the speed of the tangential image of O with respect to the arc of the curve described by O. The sign of κ depends on the relation between order along the original curve and order along the indicatrix, κ being positive if precedence corresponds to precedence. If no other circumstances influence the choice, it is natural to give the points on the indicatrix the order which makes κ positive, but if the direction of the principal normal is otherwise decided, the sign of κ is both determinate and significant.

The course taken in plane geometry is to define the standard direction of the normal as making a *positive* right angle with that of the tangent, when the sign of the curvature determines on which side of the tangent the curve is to be found. Quite analogous is the case of a geodesic on any bifacial surface: the principal normal of a geodesic must lie along the normal to the surface, and if to the former of these rays is assigned the direction of the latter, it becomes possible to express the curvature in direction and amount in the simple assertion that the curvature of the curve is the normal curvature of the surface. Another example may be drawn from the discussion of Bertrand curves—pairs of curves with a common principal normal; strictly speaking, the assumption that curvature must be positive renders the investigation usually given purely tentative, and the work would be misleading if the assumption were anything but gratuitous.

For kinematical geometry the osculating plane is the plane OTP with the cyclic direction determined by the convention that there is a positive right angle from the tangent OT to the principal normal OP. The spatial convention distinguishes one of the two directions at right angles to this plane as the positive normal direction, and it is this which is the direction of the binormal OB; in other words, OB is at right angles to both OT and OP, and there is a positive right angle from OT to OP round OB. The binormal indicatrix is traced by the point B at unit distance from Ω in the direction OB.

The torsion of the curve measures the spin of the plane OTP or the velocity of the point B with respect to the arc. Although the arguments for supposing torsion essentially positive are neither weaker nor stronger than in the case of curvature, it has long been recognised that torsion has a significant sign, the reason for the perception being that the direction in which torsion is to be measured can be assigned by a single universal convention. If the spin of the plane OTP were not wholly about OT, the tangential image T would have a velocity at right angles to this plane in addition to its velocity in the direction

OP, and this would contradict the definition of the principal normal. Hence it is always about the tangent that the osculating plane is spinning, and always parallel to the principal normal that the binormal image is moving: the torsion can therefore be given a definite sign either by the choice of *OT* rather than its reverse as the direction round which to measure the spin of *OTP*, or by the choice of *OP* rather than its reverse as the direction in which to measure the velocity of B. The two plans give opposite signs to the torsion, but unfortunately both have been used, so that care is necessary in the comparison of formulae by different writers; the first convention is now accepted as the better, and it is the spin of *OTP* about *OT* which, denoted by s, is called the torsion. But to adopt either convention in preference to postulating that torsion must be positive is to admit that the current point of a binormal indicatrix must sometimes be held to have a negative speed.

ERIC H. NEVILLE.

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29. *G.* . . . I found (Newton's *Principia*) extremely difficult, and certainly did not understand it till returned to it some time after, when I studied that wonderful work with great assiduity, and wrote numerous notes and observations upon it. I obtained a loan of what I believe was called the Jesuit's edition, which helped me. At this period mathematical science was at a low ebb in Britain; reverence for Newton had prevented men from adopting the "Calculus," which had enabled foreign mathematicians to carry astronomical and mechanical science to the highest perfection. Professors Ivory and De Morgan had adopted the "Calculus"; but several years elapsed before Mr. Herschel and Mr. Babbage were joint-editors with Professor Peacock in publishing an abridged translation of La Croix's *Treatise on the Differential and Integral Calculus*. I became acquainted with Mr. Wallace, who was, if I am not mistaken, mathematical teacher of the Military College at [Great] Marlow,* and editor of a mathematical journal published there. I had solved some of the problems contained in it and had sent them to him, which led to a correspondence, as Mr. Wallace sent me his own solutions in return. Mine was sometimes right and sometimes wrong, and it occasionally happened that we solved the same problem by different methods. At last I succeeded in solving a prize problem! It was a diophantine problem, and I was awarded a silver medal cast on purpose with my name, which pleased me exceedingly. Mr. Wallace was elected Professor of Mathematics in the University of Edinburgh,† and was very kind to me. When I told him that I earnestly desired to go through a regular course of mathematics and astronomical science, even including the highest branches, he gave me a list of the requisite books, which were in French, and consisted of Francoeur's pure *Mathematics*, and his *Elements of Mechanics*, La Croix's *Algebra*, and his large work on the *Differential and Integral Calculus*, together with his work on *Finite Differences and Series*, Biot's *Analytical Geometry and Astronomy*, Poisson's *Treatise on Mechanics*, La Grange's *Theory on Analytical Functions*, Euler's *Algebra*, Euler's *Isoperimetrical Problems* (in Latin), Clairault's *Figure of the Earth*, Monge's *Application of Analysis to Geometry*, Callet's *Logarithms*, La Place's *Mécanique Céleste*, and his *Analytical Theory of Probabilities*, etc.—M. S.

30. *H.* In writing [*Molecular and Microscopic Science*] I made a great mistake, and repent it. Mathematics are the natural bent of my mind. If I had devoted myself exclusivlely to that study, I might probably have written something useful, as a new era had begun in that science. Although I got *Chales on the Higher Geometry*, it could be but a secondary object while I was engaged in writing a popular book. Subsequently, it became a source of deep interest and occupation to me.

[This is probably Chasles, *La Géométrie Supérieure*. She wrote the above when she was between 85 and 92.]—M. S.

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* Afterwards moved to Sandhurst. †[1819].

MATHEMATICAL NOTES.

534. [K¹. δ. a.] In the preface to the seventh book of Pappus' *Mathematical Collections*, in the section on the three books of Euclid's Porisms, occur these words:

ἐὰν ὑπτίου ἢ παρυπτίου τρία τὰ ἐπὶ μιᾶς σημεῖα [ἢ παραλλήλου ἕτερα τὰ δύο] δεδομένα ᾖ, τὰ δὲ λοιπὰ πλὴν ἑνὸς ἅπτηται θέσει δεδομένης εὐθείας, καὶ τοῦθ' ἅψεται θέσει δεδομένης εὐθείας.

Many mathematicians, Fermat and Halley amongst them, tried to find out what they meant, but without success, until R. Simson in 1723, by a sort of inspired guess (see his *Opera quædam reliqua*), suggested that the words ὑπτίου and παρυπτίου denoted four-sided figures, the latter with, and the former without a re-entrant angle. No reference to these meanings is given by Liddell and Scott, and perhaps a few words of explanation may be of interest to the readers of the *Math. Gazette*.

ὕπτιος is a word of common occurrence, and means extended at full length, *supinus*. Applying this with reference to the quadrilateral *ABCD* in Fig. 1, the sides *BC*, *CD* are drawn away from the sides *AB*, *AD*.

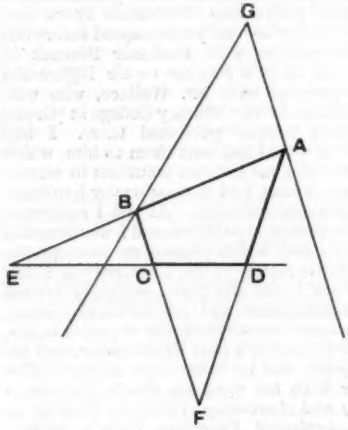

FIG. 1.

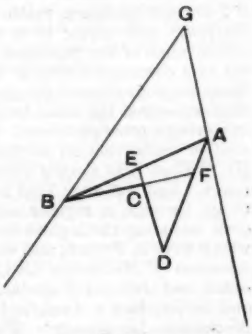

FIG. 2.

With regard to the word παρύπτιος, which is not found at all in L. & S., giving to παρά its usual meaning in compounds "alongside of," Fig. 2 gives us the sides *BC*, *CD* alongside of the sides *AB*, *AD*. The latter word was no doubt coined by Euclid, and constantly used in his lost Books of Porisms.

The above passage may be translated as follows:

Given a complete quadrilateral, with or without a re-entrant angle, if three of the points of intersection lying on one of the sides (or two of them in the case of parallelism) are fixed, and two of the remaining three move along given straight lines, the third also will move along a given straight line.

In both figures, *C*, *D*, *E* are the three fixed points of intersection, *A* and *B* move along the fixed lines *GA*, *GB*, and the proposition states that

the point F will describe a straight line, which can easily be shown to pass through G.

The particular case of parallelism referred to is of course when E is at infinity, and therefore AB parallel to CD.

The modern form in which this, the only surviving one of Euclid's Porisms, usually appears is: "coaxial triangles are copolar," a proposition generally supposed to be of quite recent discovery. JOHN J. MILNE.

535. [K¹. 21. b.] *Geometrical Construction for the Trisection of an Angle to any required Degree of Accuracy.*

[For the main idea and for a considerable share in the calculation involved, I am indebted to Cadets J. Y. G. Torlesse and M. H. C. Young, R.N.]

Let AB be an arc of a circle subtending an angle θ at the centre O.

Let C be the mid-point of the arc AB, D the mid-point of the arc BC, E the mid-point of the arc CD, F the mid-point of the arc DE, etc.

1st approximation. Let the straight line through C parallel to OA meet the straight line through D parallel to OB in Q_1.

Then $B\hat{O}Q_1 = \phi_1 =$ approximately $\frac{\theta}{3}$.

2nd approximation. Let the straight line through D parallel to OB meet the straight line through E parallel to OC in Q_2.

Let $B\hat{O}Q_2 = \phi_2$.

3rd approximation. Let the straight line through E parallel to OC meet the straight line through F parallel to OD in Q_3.

Let $B\hat{O}Q_3 = \phi_3$, etc.

Then if ϕ_n be the angle obtained in the nth approximation (*i.e.* $B\hat{O}Q_n$), it is easy to show that

$$\tan \phi_n = \frac{\Sigma \sin \alpha}{\Sigma \cos \alpha},$$

where $\alpha = \frac{\theta}{3}[1 + 8(-\frac{1}{2})^{n+1}]$, $\beta = \frac{\theta}{3}[1 - 7(-\frac{1}{2})^{n+1}]$, $\gamma = \frac{\theta}{3}[1 - (-\frac{1}{2})^{n+1}]$,

or

$$\tan\left(\phi_n - \frac{\theta}{3}\right) = \frac{\Sigma \sin u}{\Sigma \cos u},$$

where $u = \frac{8\theta}{3}(-\frac{1}{2})^{n+1}$, $v = -\frac{7\theta}{3}(-\frac{1}{2})^{n+1}$, $w = -\frac{\theta}{3}(-\frac{1}{2})^{n+1}$,

(so that $\Sigma u = 0$),

$$i.e. \quad \phi_n = \frac{\theta}{3} + \tan^{-1}\frac{\Sigma \sin u}{\Sigma \cos u}$$

$$= \frac{\theta}{3} + f(\theta),$$

where $f(\theta) = \frac{28}{3}\left(\frac{\theta}{6}\right)^3 \times (-\frac{1}{2})^{3n} + 133\left(\frac{\theta}{6}\right)^5 \times (-\frac{1}{2})^{5n} + \frac{(133)^2}{10}\left(\frac{\theta}{6}\right)^7 \times (-\frac{1}{2})^{7n} + \ldots$.

Thus, if the nth approximation gives an error of any given size in the trisection of a given angle θ, the $(n+1)$th approximation will give this same error (with the sign changed) for an angle 2θ, and approximately (rather less than) $\frac{1}{8}$th of this error for the same angle θ.

In particular, an error of less than 1 minute is given by the nth approximation if the angle θ is less than $2^n \times 10\frac{1}{2}°$, and an error of less than 10 minutes is given by the nth approximation if the angle θ is less than $2^n \times 22\frac{1}{2}°$.

D. F. FERGUSON.

REVIEWS.

Dynamics. Part I. By R. C. FAWDRY, M.A. Pp. 355. 2s. 6d. 1919. (Bell and Sons.)

No boy can use, nor master teach from Mr. Fawdry's text-books on mechanics without feeling that the subject is one of intense practical interest: we offer a warm welcome to his concluding volume on Dynamics. It is amazing within how small a compass so many and diverse applications are included, and that without any sacrifice of clearness to brevity. Free use is made of calculus methods, but the analysis is of a simple character and under modern conditions; this should not itself be a source of difficulty. The text is good, but it is the choice of subjects and illustrations and the fine quality of the examples that should ensure success and cause it to promote, as in our opinion it will, the best intersets of mathematical education.

Lecture Notes on Light. By J. R. ECCLES. Pp. 217. 12s. 6d. 1918. (Cam. Univ. Press.)

It can scarcely be questioned that there are comparatively few boys who are capable of taking intelligent and adequate notes and at the same time absorbing what is being said to them. The teacher is therefore faced with two alternatives; either he must spend valuable time in dictating notes to his pupils or he must rely either on current text-books or on notes which he may have had specially printed. Either plan has its disadvantages. The ability to take good notes is acquired only with practice, and is invaluable in University work; but the notes a boy takes unaided are incomplete and often inaccurate. We agree with Mr. Eccles in regarding a printed note-book as more useful than a text-book, although the latter may be required to supplement the former; and if printed notes are required, this note-book will be found well worthy of consideration: it is the work of an experienced, stimulating and successful teacher. C. DURELL.

The Elements of Non-Euclidean Plane Geometry and Trigonometry. By H. S. CARSLAW. Pp. xii, 179. 5s. net. 1916. (London: Longmans, Green and Co.)

This volume of "Longmans' Mathematical Series," seems to have been written in response to "recent changes in the teaching of geometry in England and America" (p. v.), which have led to a requirement for some knowledge of the hypotheses on which Euclidean geometry is built. The book is founded throughout on the historical development—as given, in particular, in Bonola's work—and forms, partly on this account, a very stimulating and instructive part of a mathematical course. It is worthy of note that non-Euclidean geometry has formed the subject in recent years of two text-books in English: that of Dr. Sommerville and this one; and both are excellent expositions on historical lines.

The first two chapters discuss the chief attempts to prove Euclid's parallel postulate (Saccheri, Legendre, Gauss, and others), and the work of the founders of the non-Euclidean geometries (Bolyai, Lobachevski, Riemann). The third and fourth chapters are respectively on hyperbolic plane geometry and hyperbolic plane trigonometry (Bolyai, Lobachevski); in the third chapter, no use is made of the principle of continuity. The fifth chapter is on measurements of length and area with the aid of the infinitesimal calculus. In the sixth and seventh chapters a treatment is given of the elliptic plane geometry and the elliptic plane trigonometry. In the eighth and last chapter, on the consistency of the non-Euclidean geometries, the advanced analytical proofs are explicitly rejected (p. 154) and Poincaré's representation of the non-Euclidean geometries by the geometry of the family of circles orthogonal or diametral to a fixed circle is given.

The only criticism that I have to offer is that Dedekind's treatment of irrational *numbers* does not depend upon any *geometrical* axiom (cf. p. 4).

It is a very pleasant sign of the times and the ability of our writers that teaching should be founded so largely on history. PHILIP E. B. JOURDAIN.

Metodi di Geometria Descrittiva. By G. LORIA. Pp. 353. 1919. (Hoepli, Milan.)

We welcome the second edition of this text-book, which is a companion and introductory volume to the author's well-known work on the practical solid geometry of curves and surfaces, published in 1912. It contains a comprehensive account of the theory and representation of points, lines and planes and of general projection. No exercises are given, but the number of special problems discussed is so large that anyone who studies Professor Loria's two volumes cannot fail to acquire a thorough knowledge of the principles of Monge and their later developments. J. J. MILNE.

The Philosophy of Mr. B*rtr*nd R*ss*ll, with an Appendix of Passages from Certain other Works. Edited by PHILIP E. B. JOURDAIN. Pp. 96. 3s. 6d. net. 1918. (London: George Allen and Unwin, Ltd. Chicago: Open Court Publishing Company.)

This is a subtle and amusing satire on the philosophy of Mr. Bertrand Russell, written from the point of view of the Red Queen in *Alice* that "Even a joke should have some meaning." The history of the fictitious "Mr. B*rtr*nd R*ss*ll" is the subject of an exquisite "Editor's Note," and the chapters, the style, and the references are all delicate parodies of Mr. Russell's work. The whole thing is elaborately done with the art that is necessary to conceal art. Of course *Through the Looking-Glass* is one of the chief sources of inspiration and illustration, and very good fun is poked at philosophers (pp. 5, 11, 13, 19, 21, 32, 47, 66, 70-3, 82, 86), politicians (pp. 17, 42), maid-servants (p. 15), valets (p. 24), literary men (pp. 16, 40-2), mathematicians (pp. 22, 25, 42, 61-2, 63, 74, 75-8), judges (pp. 17, 42, 51, 54), the great middle-class (pp. 18, 27), and Mr. Russell himself—with his absurd views on history (p. 12) and his occasional lapses from humour (p. 81 *n.*) and logic (p. 87). The splendid jokes made by De Morgan in his serious communications are given a large place, and it may be noticed that the whole thing has been accepted as a serious exposition of Mr. Russell's philosophy in a recent learned Dutch publication. M. M.

(i) Easy Numerical Trigonometry of the Right-Angled Triangle. By H. S. CARSLAW, Sc.D. Pp. 96. 2s. 6d. (Angus and Robertson.)

(ii) Easy Numerical Trigonometry. By H. S. CARSLAW, Sc.D. Pp. 142. (Angus and Robertson.)

(iii) Numerical Trigonometry. By P. ABBOTT, B.A. Pp. 163+33. 5s. 1919. (Longmans, Green and Co.)

(iv) Mathematical Tables and Formulae. By P. ABBOTT, B.A. Pp. 58. 2s. 1919. (Longmans, Green and Co.)

The recognition of the advantages of the inclusion at an early stage of some type of trigonometry in the school course is responsible for the many admirable text-books now available. Dr. Carlsaw in his introductory volumes contemplates a fairly comprehensive course before the pupil has learnt the use of logarithms, and includes angles of unlimited size and simple identities. Mr. Abbott's volume is on rather a larger scale, and contains a thorough treatment of the general solutions of triangles; his examples are numerous and of concrete interest, especial attention being directed to practical problems in solid geometry. Circular measure receives rather curt treatment in an appendix; there are many teachers who regard its discussion as stimulating and enlightening and who regret that so few of the elementary books give it fair play.

We commend Mr. Abbott's book of Tables to those who are looking for a compact volume at a reasonable price.

Solid Geometry. By R. S. HEATH, D.Sc. Pp. 123. 4s. 1919. (Rivingtons.)

A compact account of the elementary theory of lines and planes, with proofs of the chief formulae connected with the mensuration of solids and surfaces, and in particular the sphere. The proofs are set out clearly and concisely: there is an adequate supply of numerical and theoretical exercises.

Œuvres de G.-H. Halphen. Tome II. Edited by C. JORDAN, H. POINCARÉ, É. PICARD. Pp. 560. 1919. (Gauthier-Villars.)

There are few, if any, scholars who are not acquainted with Halphen's treatise on elliptic functions; his work on Mechanics and Geometry is perhaps less familiar to English students. Although he died at the comparatively early age of forty-five, he has left behind large contributions to analysis and geometry. The editors of this volume deserve our gratitude, and have set up a worthy memorial to Halphen's genius by collecting the more important papers he contributed to various scientific journals. Of papers of especial interest in the volume before us, we may mention as particularly representative of his work that entitled "Sur les singularités des courbes gauches," and the thesis submitted to the Faculté des Sciences, Paris (1878), "Sur les invariants différentiels," followed two years later by his paper, "Sur les invariants différentiels des courbes gauches."

There is a brief introductory memoir by M. C. Hermite. C. V. D.

REPORT OF SYDNEY BRANCH FOR 1918.

During the year, the secondary schools of N.S. Wales were circularised with a view to inducing the Mathematics teachers to join the Association. As a result, 21 new asssociates were obtained. There are now 46 associates and 18 members.

Two meetings were held during the year. On July 26, Mr. R. J. Lyons spoke on "Rigour in Elementary Mathematics." At the Annual Meeting on Nov. 16, Dr. E. F. Simonds spoke on "Mathematical Teaching in American High Schools." Early in the year, Mr. F. G. Brown found that he was unable to continue as Joint Hon. Sec. At the Annual Meeting in November, Mr. Wooldridge, who has been Hon. Treasurer ever since the Branch was formed, resigned. The resignation in each case was accepted with regret, both officers having given valuable services to the Association. The office-bearers for 1919 were elected as follows:—President, Prof. H. S. Carslaw; Hon. Treas., Dr. E. F. Simonds; Joint Hon. Secs., Miss F. Cohen; Mr. H. J. Meldrum. On behalf of the Executive, H. J. Meldrum, Hon. Sec.

THE LIBRARY.

CHANGE OF ADDRESS.

THE Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

SCARCE BACK NUMBERS.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them:

Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

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